

Ergodic Theory - Week 8

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1 Classifying measure preserving systems

P1. (a) Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. Show that T^k is weak mixing if and only if T is weak mixing.

Solution: ($\implies$) If T is weak mixing then for $A, B \in \mathcal{B}$ we have

$$\begin{aligned} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-kn}A \cap B) - \mu(A)\mu(B)| &\leq \frac{1}{N} \sum_{n=0}^{k(N-1)} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| \\ &= \frac{k(N-1)}{N} \cdot \frac{1}{k(N-1)} \sum_{n=0}^{k(N-1)} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| \rightarrow 0, \end{aligned}$$

as $N \rightarrow +\infty$ and, thus, T^k is weak mixing.

($\impliedby$) If T^k is weak mixing then for $A, B \in \mathcal{B}$ and $i \in \{0, \dots, k-1\}$ we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-kn}T^{-i}A \cap B) - \mu(T^{-i}A)\mu(B)|,$$

which can be rewritten as

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-(kn+i)}A \cap B) - \mu(A)\mu(B)|.$$

In particular

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| \\ &= \lim_{N \rightarrow \infty} \sum_{i=0}^{k-1} \frac{1}{N} \sum_{n \leq (N-2)/k} |\mu(T^{-(nk+i)}A \cap B) - \mu(A)\mu(B)| \\ &= \sum_{i=0}^{k-1} \lim_{N \rightarrow \infty} \frac{(N-2)}{kN} \cdot \frac{1}{(N-2)/k} \sum_{n \leq (N-2)/k} |\mu(T^{-(nk+i)}A \cap B) - \mu(A)\mu(B)| \\ &= \sum_{i=0}^{k-1} 0 = 0. \end{aligned}$$

Thus, T is weak-mixing.

- (b) Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{A}, \nu, S)$ be measure-preserving systems. Show that the system $(X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \otimes \nu, T \times S)$ is weak mixing if and only if both $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{A}, \nu, S)$ are weak mixing.

($\Leftarrow$) Since sets of the form $A \times B$ generate the product σ -algebra, it suffices to show weak mixing for sets of this form. Therefore, is enough to prove that for $A, C \in \mathcal{B}$ and $B, D \in \mathcal{A}$ there is a subset $E \subseteq \mathbb{N}$ of zero density, such that

$$\lim_{\substack{n \rightarrow \infty \\ n \in E^c}} \mu \times \nu((T \times S)^{-n}(A \times B) \cap (C \times D)) = \mu \times \nu((A \times B)) \cdot \mu \times \nu(C \times D).$$

Since X and Y are weakly mixing, there are sets E_1 and E_2 of zero density such that

$$\lim_{\substack{n \rightarrow \infty \\ n \in E_1^c}} \mu(T^{-n}A \cap C) = \mu(A)\mu(C), \text{ and } \lim_{\substack{n \rightarrow \infty \\ n \in E_2^c}} \nu(T^{-n}B \cap D) = \nu(B)\nu(D).$$

Consider $E = E_1 \cup E_2$. Notice that E has zero density. It follows that

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ n \in E^c}} \mu \times \nu((T \times S)^{-n}(A \times B) \cap (C \times D)) &= \lim_{\substack{n \rightarrow \infty \\ n \in E^c}} \mu(T^{-n}A \cap C) \nu(T^{-n}B \cap D) \\ &= \lim_{\substack{n \rightarrow \infty \\ n \in E^c}} \mu(T^{-n}A \cap C) \lim_{\substack{n \rightarrow \infty \\ n \in E^c}} \nu(T^{-n}B \cap D) \\ &= \mu(A)\mu(C)\nu(B)\nu(D) \\ &= \mu \times \nu((A \times B)) \cdot \mu \times \nu(C \times D) \end{aligned}$$

($\Rightarrow$) For this direction, it is enough to see that any factor of a weak mixing system is weak mixing. For this, suppose that $(X, \mathcal{B}, \mu, T)$ is weak mixing and that $\pi : (X, \mathcal{B}, \mu, T) \rightarrow (Y, \mathcal{A}, \nu, S)$ is a factor. Let $A, B \in \mathcal{A}$, and consider $\pi^{-1}(A), \pi^{-1}(B) \in \mathcal{B}$. Using that X is weak mixing we have that there is a set E of density zero, such that

$$\lim_{\substack{n \rightarrow \infty \\ n \in E^c}} \mu(T^{-n}\pi^{-1}(A) \cap \pi^{-1}(B)) = \mu(\pi^{-1}(A))\mu(\pi^{-1}(B)).$$

Using that $\pi\mu = \nu$ and $\pi T = S$ we get

$$\lim_{\substack{n \rightarrow \infty \\ n \in E^c}} \nu(S^{-n}A \cap B) = \nu(A)\nu(B),$$

and thus the factor is weak mixing as well.

- P2.** Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system. Prove that if the system is weak mixing then for any set $A \in \mathcal{B}$ we have

$$\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{NM} \sum_{n=1}^N \sum_{m=1}^M \mu(A \cap T^{-n}A \cap T^{-m}A \cap T^{-n-m}A) = \mu(A)^4.$$

We assume that the system is weak mixing.

We fix N and study the expression

$$\lim_{M \rightarrow \infty} \frac{1}{NM} \sum_{n=1}^N \sum_{m=1}^M \mu(A \cap T^{-n}A \cap T^{-m}A \cap T^{-n-m}A)$$

Since the system is weak mixing, we have that, for all $1 \leq n \leq N$,

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M \mu(A \cap T^{-n} A \cap T^{-m} A \cap T^{-n-m} A) = \mu(A \cap T^{-n} A) \mu(A \cap T^{-n} A) = (\mu(A \cap T^{-n} A))^2$$

Therefore, it suffices to show that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} (\mu(A \cap T^{-n} A))^2 = (\mu(A))^4 \quad (1)$$

Since the system is weak-mixing, we know that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \left| \mu(A \cap T^{-n} A) - (\mu(A))^2 \right| = 0.$$

Observe that

$$\left| (\mu(A \cap T^{-n} A))^2 - (\mu(A))^4 \right| \leq 2 \left| \mu(A \cap T^{-n} A) - (\mu(A))^2 \right|.$$

Hence,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \left| (\mu(A \cap T^{-n} A))^2 - (\mu(A))^4 \right| = 0$$

Since

$$\left| \frac{1}{N} \sum_{1 \leq n \leq N} (\mu(A \cap T^{-n} A))^2 - (\mu(A))^4 \right| \leq \frac{1}{N} \sum_{1 \leq n \leq N} \left| (\mu(A \cap T^{-n} A))^2 - (\mu(A))^4 \right|$$

by the triangle inequality, (1) follows.

P3. Let (X, A, μ, T) and (Y, B, ν, S) be two measure-preserving maps.

- (a) Show that $T \times S$ has discrete spectrum if and only if both T and S have discrete spectrum.

If $T \times S$ has discrete spectrum, there is an orthonormal basis $\{h_n\}_{n \in \mathbb{N}}$ of eigenfunctions on $X \times Y$. Called π_X and π_Y the natural projections onto X and Y respectively. We consider the functions $y \rightarrow \{\int_X h_n(x, y) d\mu(x)\}_n$ and $x \rightarrow \{\int_Y h_n(x, y) d\nu(y)\}_n$ on the spaces Y and X respectively. We will construct an orthonormal basis of eigenfunctions on X and Y using those functions. We do this for X , since the same proof works for Y . Denote $f_n(x) = \int_Y h_n(x, y) d\nu(y)$.

- f_n is an eigenfunction: Let α_n the eigenvalue of h_n . Then,

$$f_n(Tx) = \int_Y h_n(Tx, y) d\nu(y) = \int_Y h_n(Tx, Sy) d\nu(y) = \alpha_n \int_Y h_n(x, y) d\nu(y)$$

where we used the fact that S preserves ν in the middle equality.

- linear combinations of f_n are dense in $L^2(X)$: Let $f \in L^2(X)$ and consider the function $f \times 1 \in L^2(X \times Y)$. We know that for $\epsilon > 0$ there is a linear combination $\sum_{k=0}^K c_k h_k$ such that $\|\sum_{k=0}^K c_k h_k - f \times 1\|_{L^2(X \times Y)} \leq \epsilon$. In particular, using the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\left\| \sum_{k=0}^K c_k f_k - f \right\|_{L^2(X)}^2 &= \int_X \left| \int_Y \left(\sum_{k=0}^K c_k h_k(x, y) - f(x) \right) d\nu(y) \right|^2 d\mu(x) \\
&\leq \int_X \int_Y \left| \sum_{k=0}^K c_k h_k(x, y) - f(x) \right|^2 d\nu(y) d\mu(x) \\
&= \left\| \sum_{k=0}^K c_k h_k(x, y) - f \times 1 \right\|_{L^2(X \times Y)}^2 \leq \epsilon.
\end{aligned}$$

- Constructing an orthonormal subset: This collection of eigenfunctions is not necessarily orthogonal. However, as they are eigenfunctions, they are orthogonal if they correspond to different eigenvalues. So, we need to modify functions corresponding to the same eigenspace.

For an eigenvalue $\alpha \in \mathcal{S}^1$, let $\{f_{n_k}\}_{k \in \mathbb{N}}$ the set of eigenfunctions associated to α (possibly a finite set). Define $\tilde{f}_{n_1} = f_{n_1}$. Assume we have defined $\tilde{f}_{n_j}$ for all $j < k$. Call $P_k = \sum_{j < k} \tilde{f}_{n_j} \langle f_{n_k}, \tilde{f}_{n_j} \rangle$ and define

$$\tilde{f}_{n_k} = \begin{cases} (f_{n_k} - P_k) / \|f_{n_k} - P_k\| & \text{if } f_{n_k} \neq P_k \\ 0 & \text{otherwise.} \end{cases}$$

With this, after eliminating the zero functions, we have that $\{\tilde{f}_n\}_n$ is an orthonormal basis of eigenfunctions, and thus T has discrete spectrum.

Now suppose that T and S have discrete spectrum, with $\{f_n\}_n$ and $\{g_n\}_n$ orthonormal basis of eigenfunctions respectively. Consider $H = \{h_{n,m} := f_n \times g_m\}_{n,m \in \mathbb{N}}$. We claim that H is an orthonormal basis of eigenvalues of $X \times Y$. Indeed:

- Eigenfunction: Let α_n and β_m the eigenvalues of f_n and g_m . Then

$$(T \times S)h_n = T f_n \cdot S g_m = \alpha_n \beta_m f \cdot g = \alpha_n \beta_m h.$$

- Orthonormal: Let $(n_1, m_1), (n_2, m_2) \in \mathbb{N}^2$

$$\langle h_{n_1, m_1}, h_{n_2, m_2} \rangle = \int_X f_{n_1} f_{n_2} d\mu \cdot \int_Y g_{m_1} g_{m_2} d\nu = \begin{cases} 1 & \text{if } (n_1, m_1) = (n_2, m_2) \\ 0 & \text{otherwise} \end{cases}.$$

- Basis: As $\{f_n\}_n$ is dense in $L^2(X)$ and $\{g_m\}_m$ is dense in $L^2(Y)$ we have that the closure of H contains the set $\{f \times g \in L^2(X \times Y) \mid f \in L^2(X), g \in L^2(Y)\}$, which we know is dense in $L^2(X \times Y)$. Therefore $\overline{H} = L^2(X \times Y)$ and the proof is complete.

(b) Give an example of a system that is neither weak mixing nor has discrete spectrum.

Consider a non trivial weak mixing system $(X, \mathcal{A}, \mu, T)$ and a non trivial system $(Y, \mathcal{B}, \nu, S)$ with discrete spectrum. Then, the product system $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu, T \times S)$ is neither weak mixing nor has discrete spectrum. Indeed, if we assume otherwise, namely that the product system is weak mixing or that has discrete spectrum, then we will have that $(Y, \mathcal{B}, \nu, S)$ is weak mixing or that $(X, \mathcal{A}, \mu, T)$ has discrete spectrum respectively (by the previous part of this problem). Without loss of generality, assume that $(X, \mathcal{A}, \mu, T)$ has

discrete spectrum. Thus we found a system both weak mixing and with discrete spectrum. We conclude that all eigenfunctions are constant, and therefore $L^2(X)$ consists only of constant functions. This means that $(X, \mathcal{A}, \mu, T)$ is conjugated to the trivial system, which is a contradiction.

P4. Here, we study some ergodic theorems along subsequences under strong assumptions on the system (like weak-mixing).

(a) **(Optional)** Prove van der Corput's lemma: let $\mathcal{H}$ be a Hilbert space and let $(u_n)_{n \in \mathbb{N}}$ be a sequence of vectors with $\|u_n\| \leq 1$. Show that if

$$\lim_{H \rightarrow +\infty} \frac{1}{H} \sum_{0 \leq h \leq H} \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_{n+h}, u_n \rangle \right| = 0,$$

then

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{1 \leq n \leq N} u_n \right\| = 0$$

We prove the quantitatively precise and more useful estimate

$$\left\| \frac{1}{N} \sum_{1 \leq n \leq N} u_n \right\|^2 \leq \frac{C_1}{H} \sum_{1 \leq h \leq H} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_{n+h}, u_n \rangle \right| + \frac{C_2}{H} + \frac{C_3 H}{N}$$

valid for any $1 \leq H \leq N$ and some absolute constants C_1, C_2, C_3 that will be calculated throughout the proof (the constants are unimportant in general). The statement follows easily by taking $N \rightarrow +\infty$ and then $H \rightarrow +\infty$.

For each $h \in [1, H] \cap \mathbb{Z}$, we have

$$\left\| \frac{1}{N} \sum_{1 \leq n \leq N} u_n - \frac{1}{N} \sum_{1 \leq n \leq N} u_{n+h} \right\| \leq \frac{h}{N} \leq \frac{H}{N}$$

and averaging over all $1 \leq h \leq H$, we get that

$$\left\| \frac{1}{N} \sum_{1 \leq n \leq N} u_n - \frac{1}{H} \sum_{1 \leq h \leq H} \frac{1}{N} \sum_{1 \leq n \leq N} u_{n+h} \right\| \leq \frac{H}{N} \quad (2)$$

Applying the triangle inequality and then Cauchy-Schwarz, we conclude

$$\begin{aligned} \left\| \frac{1}{H} \sum_{1 \leq h \leq H} \frac{1}{N} \sum_{1 \leq n \leq N} u_{n+h} \right\|^2 &= \left\| \frac{1}{N} \sum_{1 \leq n \leq N} \frac{1}{H} \sum_{1 \leq h \leq H} u_{n+h} \right\|^2 \leq \\ &\left(\frac{1}{N} \sum_{1 \leq n \leq N} \left\| \frac{1}{H} \sum_{1 \leq h \leq H} u_{n+h} \right\| \right)^2 \leq \frac{1}{N} \sum_{1 \leq n \leq N} \left\| \frac{1}{H} \sum_{1 \leq h \leq H} u_{n+h} \right\|^2 \end{aligned} \quad (3)$$

We expand out the right-hand side as

$$\frac{1}{N} \sum_{1 \leq n \leq N} \frac{1}{H^2} \sum_{1 \leq h_1, h_2 \leq H} \langle u_{n+h_1}, u_{n+h_2} \rangle$$

The diagonal contribution $h_1 = h_2$ is equal to

$$\frac{1}{N} \sum_{1 \leq n \leq N} \frac{1}{H^2} \sum_{1 \leq h \leq H} \|u_{n+h}\|^2 \leq \frac{1}{H},$$

since all terms are bounded in magnitude by 1.

For the off-diagonal contribution, we use symmetry to get that

$$\begin{aligned} \left| \frac{1}{N} \sum_{n \leq N} \frac{1}{H^2} \sum_{1 \leq h_1, h_2 \leq H} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| &\leq \left| \frac{2}{N} \sum_{n \leq N} \frac{1}{H^2} \sum_{1 \leq h_1 < h_2 \leq H} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| = \\ &\left| \frac{2}{H^2} \sum_{1 \leq h_1 < h_2 \leq H} \frac{1}{N} \sum_{n \leq N} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| \leq \frac{2}{H^2} \sum_{1 \leq h_1 < h_2 \leq H} \left| \frac{1}{N} \sum_{n \leq N} \langle u_{n+h_1}, u_{n+h_2} \rangle \right|. \end{aligned}$$

Observe that

$$\begin{aligned} \left| \frac{1}{N} \sum_{n \leq N} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| &= \left| \frac{1}{N} \sum_{h_1+1 \leq n \leq N+h_1} \langle u_n, u_{n+h_2-h_1} \rangle \right| \leq \\ &\left| \frac{1}{N} \sum_{1 \leq n \leq N+h_1} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2h_1}{N} \leq \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2H}{N} \end{aligned}$$

where we used the triangle inequality for the first inequality and bounded trivially each term by 1.

Using this inequality, we get that the off-diagonal contributions is bounded by

$$\begin{aligned} \frac{2}{H^2} \sum_{1 \leq h_1 < h_2 \leq H} \left(\left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2H}{N} \right) &\leq \\ \frac{2}{H^2} \sum_{1 \leq h_1 < h_2 \leq H} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+h_2-h_1} \rangle \right| + \frac{2H}{N} \end{aligned}$$

The difference $h_2 - h_1$ takes values in $1, \dots, H-1$ and a simple counting yields that every r in this range occurs as the difference of $H-r$ pairs. Therefore, we may rewrite the right-hand side as

$$\begin{aligned} \frac{2}{H^2} \sum_{1 \leq r \leq H-1} (H-r) \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} &\leq \frac{2}{H} \sum_{1 \leq r \leq H-1} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} \leq \\ &\frac{2}{H} \sum_{1 \leq r \leq H} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} + \frac{2}{H}. \end{aligned}$$

Using this, as well as the contribution from the diagonal terms, we deduce from (3) the bound

$$\left\| \frac{1}{H} \sum_{1 \leq h \leq H} \frac{1}{N} \sum_{1 \leq n \leq N} u_{n+h} \right\|^2 \leq \frac{2}{H} \sum_{1 \leq r \leq H} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+r} \rangle \right| + \frac{2H}{N} + \frac{3}{H}.$$

Using (2) and the triangle inequality, we deduce that

$$\left\| \frac{1}{N} \sum_{1 \leq n \leq N} u_n \right\|^2 \leq \frac{2}{H} \sum_{1 \leq h \leq H} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle u_n, u_{n+h} \rangle \right| + 3 \left(\frac{H}{N} + \frac{1}{H} \right).$$

- (b) Use part (a) to show that if a system $(X, \mathcal{B}, \mu, T)$ is totally ergodic (that is, T^k is ergodic for all $k \in \mathbb{N}$) then for any function $f \in L^\infty(\mu)$, we have

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{n^2} f - \int f d\mu \right\|_{L^2(\mu)} = 0$$

Setting $g = f - \int f d\mu$, we have $\int g d\mu = 0$ and our problem can be rewritten as

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{n^2} g \right\|_{L^2(X)} = 0.$$

We will prove this by using part (a). Our Hilbert space is $L^2(X)$ and our sequence is $u_n = T^{n^2} g$. Without loss of generality, we can rescale g and assume that $\|g\|_{L^\infty(X)} \leq 1$ and thus $\|T^{n^2} g\| \leq 1$.

Let H be a fixed positive integer and consider the expression

$$\frac{1}{H} \sum_{1 \leq h \leq H} \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle T^{(n+h)^2} g, T^{n^2} g \rangle \right| = \frac{1}{H} \sum_{0 \leq h \leq H} \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \int T^{(n+h)^2} g \cdot \overline{T^{n^2} g} d\mu \right|$$

Because T preserves the measure μ , we can compose with T^{-n^2} in the integral to get

$$\int T^{(n+h)^2} g \cdot \overline{T^{n^2} g} d\mu = \int \bar{g} \cdot T^{2nh+h^2} g d\mu$$

.

Therefore, we can rewrite the double average as

$$\frac{1}{H} \sum_{1 \leq h \leq H} \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \int \bar{g} \cdot T^{2hn} (T^{h^2} g) d\mu \right|.$$

Since the system is totally ergodic, the transformation T^{2h} is ergodic for all $h \neq 0$ and we have that the averages

$$\frac{1}{N} \sum_{1 \leq n \leq N} T^{2hn} (T^{h^2} g)$$

converge to the integral $\int T^{h^2} g \, d\mu = \int g \, d\mu = 0$ in $L^2(X)$. The Cauchy-Schwarz inequality implies that

$$\left| \int \bar{g} \left(\frac{1}{N} \sum_{1 \leq n \leq N} T^{2hn}(T^{h^2} g) \, d\mu \right) \right| \leq \|g\|_{L^2(X)} \left\| \frac{1}{N} \sum_{1 \leq n \leq N} T^{2hn}(T^{h^2} g) \right\|_{L^2(X)} \rightarrow 0.$$

Therefore, each term in the average over h is zero and so the whole average is zero. Taking $H \rightarrow +\infty$, we conclude that the assumptions in part (a) are satisfied and our claim follows.

(c) Use part (a) to show that if $(X, \mathcal{B}, \mu, T)$ is weak-mixing, then for any $f \in L^\infty(\mu)$, we have

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{n=1}^N T^n f \cdot T^{2n} f - \left(\int f \, d\mu \right)^2 \right\|_{L^2(X)} = 0$$

Conclude that if $(X, \mathcal{B}, \mu, T)$ is weak-mixing, then for any set $A \in \mathcal{B}$, we have

$$\frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n} A \cap T^{-2n} A) = (\mu(A))^3.$$

We prove the more general statement that

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{n=1}^N T^n f \cdot T^{2n} g - \int f \, d\mu \int g \, d\mu \right\|_{L^2(X)} = 0$$

As in the previous exercise, we may write $f_1 = f - \int f \, d\mu$ and $g_1 = g - \int g \, d\mu$. Observe that

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N T^n \left(f_1 + \int f \, d\mu \right) \cdot T^{2n} \left(g_1 + \int g \, d\mu \right) - \int f \, d\mu \int g \, d\mu = \\ \frac{1}{N} \sum_{n=1}^N T^n f_1 \cdot T^{2n} g_1 - \int g \, d\mu \left(\frac{1}{N} \sum_{n=1}^N T^n f_1 \right) - \int f \, d\mu \left(\frac{1}{N} \sum_{n=1}^N T^{2n} g_1 \right). \end{aligned}$$

Our result will follow if we show that each one of the three terms converges to zero in $L^2(X)$.

The second term converges to zero in $L^2(X)$ by the mean ergodic theorem, since T is ergodic (as it is weak-mixing) and $\int f_1 \, d\mu = 0$. Similarly, the third term converges in $L^2(X)$ to zero, since T^2 is ergodic (T^2 is weak-mixing by the first exercise, therefore it is ergodic).

It remains to show that

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{n=1}^N T^n f_1 \cdot T^{2n} g_1 \right\|_{L^2(X)} = 0.$$

We will prove this using part (a). Our Hilbert space is $L^2(X)$ and our sequence $u_n = T^n f_1 T^{2n} g_1$. We may rescale f_1, g_1 and assume that they are 1-bounded, in which case $\|u_n\|_{L^2(X)} \leq 1$.

Let H be a positive integer and consider the expression

$$\begin{aligned} \frac{1}{H} \sum_{0 \leq h \leq H} \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \langle T^{n+h} f_1 \cdot T^{2(n+h)} g_1, T^n f_1 \cdot T^{2n} g_1 \rangle \right| = \\ \frac{1}{H} \sum_{0 \leq h \leq H} \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{1 \leq n \leq N} \int T^{n+h} f_1 \cdot T^{2(n+h)} g_1 \cdot \overline{T^n f_1 \cdot T^{2n} g_1} d\mu \right| \end{aligned}$$

We compose with T^{-n} in the integral above to get that

$$\int T^{n+h} f_1 \cdot T^{2(n+h)} g_1 \cdot \overline{T^n f_1 \cdot T^{2n} g_1} d\mu = \int (\overline{f_1} \cdot T^h f_1) \cdot T^n (\overline{g_1} \cdot T^{2h} g_1) d\mu.$$

Since T is ergodic, we have that

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{1 \leq n \leq N} T^n (\overline{g_1} \cdot T^{2h} g_1) - \int \overline{g_1} \cdot T^{2h} g_1 d\mu \right\|_{L^2(X)} = 0$$

by the mean ergodic theorem. Using a similar argument as in the previous part, we deduce that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{1 \leq n \leq N} \int (\overline{f_1} \cdot T^h f_1) \cdot T^n (\overline{g_1} \cdot T^{2h} g_1) d\mu &= \int (\overline{f_1} \cdot T^h f_1) \left(\int \overline{g_1} \cdot T^{2h} g_1 d\mu \right) d\mu = \\ &= \int \overline{f_1} \cdot T^h f_1 d\mu \int \overline{g_1} \cdot T^{2h} g_1 d\mu. \end{aligned}$$

Therefore, our average over h is equal to

$$\frac{1}{H} \sum_{0 \leq h \leq H} \left| \int \overline{f_1} \cdot T^h f_1 d\mu \int \overline{g_1} \cdot T^{2h} g_1 d\mu \right|$$

and we want to show that this limit is zero. Since we assume that $\|g_1\|_{L^\infty(X)} \leq 1$, we have that

$$\left| \int \overline{g_1} \cdot T^{2h} g_1 d\mu \right| \leq \int |\overline{g_1} \cdot T^{2h} g_1| d\mu \leq 1$$

and, thus, it suffices to show that

$$\lim_{N \rightarrow +\infty} \frac{1}{H} \sum_{0 \leq h \leq H} \left| \int \overline{f_1} \cdot T^h f_1 d\mu \right| = 0$$

Since the system is weak-mixing and $f_1 \in L^\infty(X)$ (so it is also square-integrable), we know that

$$\lim_{N \rightarrow +\infty} \frac{1}{H} \sum_{0 \leq h \leq H} \left| \int \overline{f_1} \cdot T^h f_1 d\mu - \int f_1 d\mu \cdot \int \overline{f_1} d\mu \right| = 0$$

and since $\int f_1 = 0$, the conclusion follows.

To get the last convergence, we apply our result for $f = g = \mathbb{1}_A$ to deduce that

$$\lim_{N \rightarrow +\infty} \left\| \frac{1}{N} \sum_{n=1}^N T^n \mathbb{1}_A T^{2n} \mathbb{1}_A - \left(\int \mathbb{1}_A d\mu \right)^2 \right\| = 0$$

and this coupled with the Cauchy-Schwarz inequality imply that

$$\lim_{N \rightarrow +\infty} \left| \int \mathbb{1}_A \left(\frac{1}{N} \sum_{n=1}^N T^n \mathbb{1}_A T^{2n} \mathbb{1}_A \right) d\mu - \left(\int \mathbb{1}_A d\mu \right)^3 \right| = 0,$$

Since $\int \mathbb{1}_A \cdot T^n \mathbb{1}_A \cdot T^{2n} \mathbb{1}_A d\mu = \int \mathbb{1}_{A \cap T^{-n}A \cap T^{-2n}A} d\mu = \mu(A \cap T^{-n}A \cap T^{-2n}A)$, the result follows.